

# A NOTE ON THE FUNDAMENTAL GROUP OF CONTACT TORIC MANIFOLDS OF REEB TYPE

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ABSTRACT. Let  $(M, \alpha)$  be a connected compact contact toric manifold of Reeb type. In this note, we give a proof that the fundamental group of  $M$  is finite cyclic. We also point out the fundamental groups of certain related quotients of  $M$ .

## 1. INTRODUCTION

Let  $(M, \alpha)$  be a co-oriented contact manifold, where  $\alpha$  is a contact one-form. Let  $T^k$  be a connected compact  $k$ -dimensional torus acting on  $M$  preserving the contact form  $\alpha$ , hence preserving the contact structure  $\xi = \ker(\alpha)$ . A particular family called contact toric manifolds of Reeb type is as follows.

**Definition 1.1.** A  $2n + 1$ -dimensional contact manifold with an effective  $T^{n+1}$  action preserving the contact structure is called a **contact toric manifold**. If the Reeb vector field of a contact form generates an  $\mathbb{R}$  subaction of the  $T^{n+1}$  action, then the contact  $T^{n+1}$ -manifold is called a **contact toric manifold of Reeb type**.

Recall that a  $2n$ -dimensional symplectic manifold with an effective  $T^n$ -action preserving the symplectic structure is called a symplectic toric manifold. Contact toric manifolds are the odd dimensional analog of symplectic toric manifolds. Compact symplectic toric manifolds and compact contact toric manifolds are both classified ([4] and [6]). For compact symplectic toric manifolds, the torus action must have fixed points, in contrast, for compact contact toric manifolds, the torus action cannot have fixed points ([6, Lemma 2.12]).

Compact contact toric manifolds of Reeb type is a special class of contact toric manifolds. They are most analogous to compact symplectic toric manifolds, in the sense that their moment map images are also simple convex polytopes ([1, 3, 6]), similar to the Atiyah, Guillemin and Sternberg convexity theorem for torus actions on symplectic manifolds. It is observed that compact contact toric manifolds of Reeb type are **K-contact** ([7, Prop. 3.1], or [5, Prop. 8]), even Sasakian manifolds ([1, Theorem 5.2]). (See Definition 3.1 for K-contact manifolds.) The study of the geometry and topology of K-contact manifolds is a very interesting topic, it attracted a great amount of attention among the people who worked on the subject with different aspects.

Let  $(M, \omega)$  be a compact symplectic toric manifold, and  $\Phi$  the moment map. It is well known that for any  $a \in \Phi(M)$ ,  $\Phi^{-1}(a)$  consists of a single  $T$ -orbit, and  $M/T$  is homeomorphic to a convex polytope  $\Phi(M)$ . Moreover,  $M$  is simply connected by

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the results in [9]. In this paper, we will see that compact contact toric manifolds are not exactly the same. [6, Theorem 2.18] gives the classification of all connected compact contact toric manifolds. By the criteria of Section 3 of the current paper, for example, we can explicitly list those which are not of Reeb type — they are torus bundles over spheres, so their topology is somehow understandable. If the manifold is of Reeb type, Lerman shows in [7] that its fundamental group is a finite abelian group. In this note, we prove that the fundamental group of a compact contact toric manifold of Reeb type is, in fact, a finite cyclic group. Moreover, we also point out the fundamental groups of certain related quotient spaces of the manifold. (Three dimensional) Lens spaces are compact contact toric manifolds of Reeb type. They provide examples of such spaces with nontrivial fundamental groups.

Our method of proof shows a very different perspective. The point of view is direct and transparent.

**Theorem 1.** *Let  $(M, \alpha)$  be a connected compact contact toric manifold of Reeb type. Then*

- (1)  $\pi_1(M)$  is a finite cyclic group,
- (2)  $\pi_1(M/T) = 1$ , and
- (3) let  $M_a = \Phi^{-1}(a)/T$ , then  $\pi_1(M_a) = 1$  for any  $a \in \Phi(M)$ , where  $\Phi$  is the contact moment map.

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## 2. CONTACT MOMENT MAP

Let  $(M, \alpha)$  be a co-oriented contact manifold with a compact Lie group  $G$  action preserving  $\alpha$ . The contact moment map  $\Phi$  is defined to be

$$\langle \Phi(x), X \rangle = \alpha_x(X_M(x))$$

for all  $x \in M$  and all  $X \in \mathfrak{g} = \text{Lie}(G)$ , where  $X_M$  is the vector field on  $M$  generated by the  $X$ -action. This moment map depends on the contact form  $\alpha$ : if  $f$  is an invariant smooth function on  $M$ , then  $\alpha' = e^f \alpha$  gives the same contact distribution  $\xi = \ker(\alpha)$ , but the moment map corresponding to  $\alpha'$  is

$$\Phi' = e^f \Phi.$$

Hence one often considers the **moment cone**

$$C(\Phi) = \{t\Phi(x) \mid t \geq 0, x \in M\},$$

which is also described as the union of  $\{0\}$  and the image of the symplectic moment map on the **symplectization**  $(M \times \mathbb{R}, d(e^t \alpha))$  of  $M$ .

For our purpose, it will be sufficient to consider the contact form dependant moment map  $\Phi$ .

Now we collect a few known results which we will use. For the following convexity result, see [1] or [3].

**Theorem 2.** *Let  $(M, \alpha)$  be a  $2n + 1$ -dimensional connected compact contact toric manifold of Reeb type. Let  $\Phi$  be the contact moment map. Then  $\Phi(M)$  is a compact convex (rational) polytope of dimension  $n$ .*

For the following result, see [6, Lemma 3.16] and [8] (or [6, Lemma 4.3]).

**Theorem 3.** *Let  $(M, \alpha)$  be a connected compact contact toric manifold. Let  $\Phi$  be the contact moment map. Then for any  $a \in \Phi(M)$ , each connected component of  $\Phi^{-1}(a)$  is a  $T$ -orbit, and if  $\dim(M) > 3$ , then  $\Phi^{-1}(a)$  consists of a single  $T$ -orbit.*

**Proposition 2.1.** *Let  $(M, \alpha)$  be a  $2n + 1$ -dimensional connected compact contact toric manifold of Reeb type. Then  $M/T$  is homeomorphic to a compact convex polytope of dimension  $n$ .*

*Proof.* By Theorems 2 and 3, if  $\dim(M) > 3$ , then  $M/T$  is homeomorphic to  $\Phi(M)$ , the compact convex polytope. If  $\dim(M) = 3$ , by [6, Theorem 2.18], 3-dimensional connected compact contact toric manifolds are either  $T^2 \times S^1$  or lens spaces. The former has contact moment map image  $S^1$ . By Theorem 2, those of Reeb type are among the lens spaces, then by [6, Lemma 6.1],  $M/T$  is homeomorphic to a connected closed interval.  $\square$

### 3. CONTACT TORIC MANIFOLDS OF REEB TYPE AND K-CONTACT MANIFOLDS

In this section, we will see that contact toric manifolds of Reeb type with dimension  $2n + 1$  are K-contact manifolds of rank  $n + 1$ . We review known results for such manifolds which we will use in the next section to prove the main theorem.

**Definition 3.1.** A **K-contact manifold** is a manifold  $M$  with a contact form  $\alpha$  and a Riemannian metric  $g$  **adapted** to  $\alpha$ , i.e.,

- (1)  $g$  is preserved by the Reeb flow of  $\alpha$ , and
- (2) there exists an almost complex structure  $J$  on  $\ker \alpha$  such that  $g(X, Y) = d\alpha(X, JY)$  for all  $X, Y \in C^\infty(\ker \alpha)$ .

For example, Sasakian manifolds are K-contact.

Let  $(M, \alpha, g)$  be a compact K-contact manifold. The closure of the Reeb flow in the isometry group  $\text{Isom}(M, g)$  is a connected compact torus  $T$ , so  $M$  admits a natural  $T$ -action. The dimension of  $T$  is called the **rank** of  $(M, \alpha, g)$ . The rank of a  $2n + 1$ -dimensional K-contact manifold is bounded from above by  $n + 1$ .

We will use the following results.

**Proposition 3.2.** [5, Proposition 8] *Any compact contact toric  $2n + 1$ -dimensional manifold of Reeb type has a K-contact structure of rank  $n + 1$  such that every closed Reeb orbit is isolated.*

**Proposition 3.3.** [10] or [5, Proposition 5] *Let  $(M, \alpha, g)$  be a compact K-contact manifold. Let  $\Phi$  be the moment map for the natural  $T$ -action on  $M$ . Then for a generic  $X \in \mathfrak{t} = \text{Lie}(T)$ , the component  $\Phi^X = \langle \Phi, X \rangle$  is a  $T$ -invariant Morse-Bott function with critical set the set of closed Reeb orbits, and each component of the critical set has even index.*

This is part of Theorem 3 in [5]:

**Proposition 3.4.** *Let  $(M, \alpha, g)$  be a compact K-contact manifold. If the closed Reeb orbits of  $\alpha$  are isolated, then  $H^1(M; \mathbb{R}) = 0$ .*

## 4. PROOF OF THEOREM 1

To prove Theorem 1, we first prove the following proposition.

**Proposition 4.1.** *Let  $(M, \alpha)$  be a connected compact contact toric manifold of Reeb type. Then  $\pi_1(M)$  is a quotient group of  $\mathbb{Z}$ , and it cannot be  $\mathbb{Z}$ .*

*Proof.* By Proposition 3.2,  $M$  admits a K-contact structure  $(\alpha', g)$  such that the closed Reeb orbits of  $\alpha'$  are isolated. Then by Proposition 3.4,  $H^1(M; \mathbb{R}) = 0$ . So  $\pi_1(M) \neq \mathbb{Z}$ .

Together with Proposition 3.3, we know that a component  $\Phi^X$  of the contact moment map for  $\alpha'$  is a Morse-Bott function with critical set the set of closed Reeb orbits of  $\alpha'$ , which are isolated circles. Since  $M$  is connected compact,  $\Phi^X(M)$  is a connected closed interval. Since the Morse index of each critical set is even and  $M$  is connected,  $\Phi^X$  has a unique minimum and a unique maximum. Let  $A$  be the minimum, and  $B$  the maximum. Let  $c_0, c_1, \dots$ , and  $c_m$  be the critical values of  $\Phi^X$ , and  $a_0, a_1, \dots, a_{m-1}$  be regular values of  $\Phi^X$  so that

$$\Phi^X(A) = c_0 < a_0 < c_1 < a_1 < \dots < a_{m-1} < c_m = \Phi^X(B).$$

Let

$$M^{a_0} = \{m \in M \mid \Phi^X(m) < a_0\}, \text{ and } M^{a_1} = \{m \in M \mid \Phi^X(m) < a_1\}.$$

Then  $M^{a_0}$  has the homotopy type of  $A$ . Hence

$$\pi_1(M^{a_0}) = \pi_1(A) = \mathbb{Z}.$$

Let  $C_1 \in (\Phi^X)^{-1}(c_1)$  be a critical component, and first assume for simplicity that  $C_1$  is the only critical component in  $(\Phi^X)^{-1}(c_1)$ . By Morse-Bott theory,  $M^{a_1}$  has the homotopy type of  $M^{a_0}$  attached by  $D^-(C_1)$ , the negative normal bundle of  $C_1$ . By the Van-Kampen theorem,

$$\pi_1(M^{a_1}) = \pi_1(M^{a_0}) *_{\pi_1(S^-(C_1))} \pi_1(D^-(C_1)),$$

where  $S^-(C_1)$  is the sphere bundle of  $D^-(C_1)$ .

First assume that  $C_1$  has index 2. Then the fiber of  $D^-(C_1)$  is a 2-disk, and the fiber of  $S^-(C_1)$  is a circle. Note that the map

$$f: \pi_1(S^-(C_1)) \rightarrow \pi_1(D^-(C_1)) = \pi_1(C_1)$$

is onto, and  $\ker(f) = \pi_1(S^1) = \mathbb{Z}$ , where  $S^1$  is the fiber of  $S^-(C_1)$ . Also note that the image of  $\ker(f) = \mathbb{Z}$  in  $\pi_1(M^{a_0}) = \mathbb{Z}$  is a subgroup of  $\mathbb{Z}$ , hence is  $k\mathbb{Z}$  for some nonnegative integer  $k$ . Hence

$$\pi_1(M^{a_1}) = \mathbb{Z}/k\mathbb{Z}.$$

Next assume that  $C_1$  has index bigger than 2. Then the fiber of  $S^-(C_1)$  is simply connected. So the corresponding map  $f$  above is an isomorphism, hence

$$\pi_1(M^{a_1}) = \pi_1(M^{a_0}).$$

If there is another critical component  $C_2$  in  $(\Phi^X)^{-1}(c_1)$ , then attach the negative normal bundle of  $C_2$  to the previous obtained space, and use the above argument again.

Inductively let  $\Phi^X$  cross the critical levels, and repeat the argument each time attaching the negative normal bundle of a critical set. The argument above shows that only when crossing a critical set of index 2 may result in a change of  $\pi_1$  of the

new space and the previous space, and  $\pi_1$  of the new space is a quotient group of  $\mathbb{Z}$  or a quotient group of a cyclic group. Hence in the end we get

$$\pi_1(M) = \mathbb{Z}_l \text{ or } \mathbb{Z}, \text{ for some } l \in \mathbb{N}.$$

and it cannot be  $\mathbb{Z}$  by the first paragraph.  $\square$

From the proof, we observe the following fact:

**Corollary 4.2.** *Let  $(M, \alpha, g)$  be a compact  $K$ -contact manifold with isolated closed Reeb orbits. Then the Morse-Bott function  $\Phi^X$  in Proposition 3.3 has at least one critical circle of index 2.*

Now we summarize the proof of Theorem 1.

*Proof of Theorem 1.* (1) follows from the proposition above, (2) follows from Proposition 2.1, and (3) follows from Theorem 3.  $\square$

## 5. THE CRITICAL SET OF THE MORSE-BOTT FUNCTION AND BASIC COHOMOLOGY

We proved our main theorem in the last section. The material in this section is not needed for proving the main theorem. We include it here for better understanding the critical set of the Morse-Bott function used in the proof of the theorem and for understanding Proposition 3.4. The idea of basic cohomology plays an important role here. We hope this section can serve as a sketch of some main ideas relative to us. The reader may wish to compare it with the case of a symplectic manifold.

Let  $(M, \alpha)$  be a contact manifold. Let  $\mathcal{F}$  be the foliation given by the flow of the Reeb vector field. A differential form  $\beta$  on  $M$  is called  **$\mathcal{F}$ -basic**, if for any vector field  $X$  tangent to the leaves of  $\mathcal{F}$ , we have  $i_X \beta = 0$  and  $L_X \beta = 0$ . The set of basic forms on  $M$  forms a differential complex, its cohomology is called the  **$\mathcal{F}$ -basic cohomology**, denoted  $H^*(M, \mathcal{F})$ .

The following fact is implied in [5]. We state it in the way which suits our purpose. The proof is by following the ideas of [5].

**Proposition 5.1.** *Let  $(M, \alpha, g)$  be a compact  $K$ -contact manifold of dimension  $2n + 1$ . Assume that the closed Reeb orbits of  $\alpha$  are isolated. Let  $\Phi^X$  be a Morse-Bott function as in Proposition 3.3. Then for each  $0 \leq 2j \leq 2n$ , there is at least one closed Reeb orbit of index  $2j$  for  $\Phi^X$ .*

*Proof.* By [5, Proposition 6], the function  $\Phi^X$  is a perfect Morse-Bott function for the  $\mathcal{F}$ -basic cohomology, where  $\mathcal{F}$  is the Reeb flow foliation. So we have

$$P_t(M, \mathcal{F}) = \sum_i t^i \dim H^i(M, \mathcal{F}) = \sum_B t^{\lambda_B} P_t(B/\mathcal{F}),$$

where the sum on the right is over the connected components  $B$ 's of the set of closed Reeb orbits,  $\lambda_B$  is the Morse index of  $B$ , which is even by Proposition 3.3,  $B/\mathcal{F}$  is the leaf space of  $B$  and  $P_t(B/\mathcal{F})$  is its Poincaré polynomial. In our case, each  $B$  is a circle and  $B/\mathcal{F}$  is a point, so  $P_t(B/\mathcal{F}) = 1$ . Hence

$$\sum_i t^i \dim H^i(M, \mathcal{F}) = \sum_j t^{2j} n_j,$$

where  $n_j$  is the number of closed Reeb orbits of index  $2j$ . Hence  $H^{\text{odd}}(M, \mathcal{F}) = 0$ . The even degree forms  $(d\alpha)^j$ , where  $0 \leq j \leq n$ , are basic differential forms representing non-zero elements of  $H^{2j}(M, \mathcal{F})$ ,  $0 \leq j \leq n$ , (due to the nondegeneracy of  $d\alpha$  on  $\ker(\alpha)$ ). Hence the claim follows.  $\square$

The idea of proof of Proposition 3.4 is as follows. From the proof above, we have  $H^1(M, \mathcal{F}) = 0$ . Now, by the Gysin sequence of the isometric flow  $\mathcal{F}$  ([11] or p215 in [2]), we have

$$0 \longrightarrow H^1(M, \mathcal{F}) \longrightarrow H^1(M) \longrightarrow H^0(M, \mathcal{F}) \xrightarrow{\cup} H^2(M, \mathcal{F}) \longrightarrow \cdots,$$

where the map  $\cup$  is multiplication by  $d\alpha$  which is injective due to the nondegeneracy of  $d\alpha$  on  $\ker(\alpha)$ . Hence  $H^1(M) = H^1(M, \mathcal{F}) = 0$ .

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